Towers, Bounding, Dominating

Joint work (very much in progress) with
Martin Goldstern, Lorenz Halbeisen, and Salome Schumacher

Asaf Karagila

Vienna University of Technology

1 February 2018

Winter School in Abstract Analysis 2018

The goal of this talk is to try and answer the following:

Question

How do we define cardinal characteristics of the continuum in ZF*, and what can we say about them?*

Cardinal Characteristics of the Continuum

The continuum is a versatile object for a set theorist. It can be ω^ω or 2^ω, or it can be any uncountable Polish space (the real numbers, $\mathbb{R}$, for example), and it can be $\mathcal{P}(\omega)$, or even $[\omega]^\omega$.

Each of these objects has a natural associated structure, from which natural cardinal invariant occur. Let us focus on these three for now:

- $\mathfrak{t}$: The smallest cardinal κ such that there is a family $\{A_\alpha \mid \alpha < \kappa\} \subseteq [\omega]^\omega$ such that $A_\beta \subseteq^* A_\alpha$, without a lower bound with respect to $\subseteq^*$.
- $\mathfrak{b}$: The smallest cardinal κ such that there is a family $\{f_\alpha \mid \alpha < \kappa\} \subseteq \omega^\omega$ such that there is no f for which $f_\alpha \leq^* f$ for all $\alpha < \kappa$.
- $\mathfrak{d}$: The smallest cardinal κ such that there is a family $\{f_\alpha \mid \alpha < \kappa\} \subseteq \omega^\omega$ such that for all f there is some α for which $f \leq^* f_\alpha$.

What goes wrong when choice fails?

So if we do not assume the axiom of choice, what is the problem with the existing definitions?

For example, in Cohen's first model of $\neg$AC there is a Dedekind-finite set of Cohen reals over L. (Recall that a Dedekind-finite set is a set that every proper subset is strictly smaller in cardinality.)

One can show that every infinite of subset of this set is in fact unbounded. And actually much more (e.g. it is a family with strong finite-intersection but no pseudo-intersection).

(Since adding a single Cohen real does not change $\mathfrak{t}$, it follows that $\mathfrak{t}$ is the same as the ground model (in this case L, so $\aleph_1$), but $\mathfrak{p}$ is not well-defined, so $\mathfrak{p} \neq \mathfrak{t}$.)

What goes wrong when choice fails?

So it is possible that the cardinals of unbounded families are not well-founded, so there is no minimal—let alone a minimum—cardinal.

Or, it is possible that there are no free ultrafilters, so $\mathfrak{u}$ is not defined; and there might be no MAD families so $\mathfrak{a}$ is not defined; and so on and so forth…

So how can we correct for that?

Changing the problem so we can solve it

Let us look at $\mathfrak{b}$ again.

Definition ($\mathfrak{b}$)

The smallest cardinal κ such that there is a family $\{f_\alpha \mid \alpha < \kappa\} \subseteq \omega^\omega$ such that there is no f for which $f_\alpha \leq^* f$ for all $\alpha < \kappa$.

This is not an invariant of ω^ω. This an invariant of $\omega^\omega/\mathrm{fin}$.

So we can just define cardinal characteristics in terms of *true* natural structures.

Redefining CCCs

So we can define the cardinal characteristics as follows:

- $\mathfrak{t}$: The smallest cardinal of a well-ordered chain in $[\omega]^\omega/\mathrm{fin}$ without a lower bound.
- $\mathfrak{b}$: The smallest cardinal of an unbounded subset of $\omega^\omega/\mathrm{fin}$.
- $\mathfrak{d}$: The smallest cardinal of a dominating family in $\omega^\omega/\mathrm{fin}$.

Assuming ZFC, as every equivalence class in $\mathcal{P}(\omega)/\mathrm{fin}$ is countable, these definitions are in fact equivalent to the "usual" ones. So we do not actually change anything in the known results.

Some motivation

Having a definition is not enough. We also want theorems.

Consider the following situation:

1. We start with some model of ZFC.
2. We iterate adding Cohen reals, one at a time a_n.
3. By clever choice of an intermediate model at the ωth stage we will have the sequence of $A_n = a_n/\mathrm{fin}$, but not the sequence of the a_n's.
4. The set $A = \bigcup A_n$ is therefore in the model, and it is an unbounded family of reals. It is therefore uncountable, but it is a countable union of countable sets.

But is A *really* uncountable? Well, yes. But the essence of A, as far as matters of bounded-ness go, are countable.

Therefore defining $\mathfrak{b}$ and other cardinals from $\omega^\omega/\mathrm{fin}$ is a good way to go about it.

Not all problems are solved...

Some problems persist even with the new definition.

1. Some cardinals are simply not well-defined.
2. It might be that there is a Dedekind-finite family of equivalence classes with certain properties.
3. It is consistent that $|\omega^\omega/\mathrm{fin}| > 2^{\aleph_0}$, so if $\mathfrak{c}$ is the cardinal of $\omega^\omega/\mathrm{fin}$, we get that $\mathfrak{c} > 2^{\aleph_0}$.

Some almost-theorems (with fairly big gaps)

"Theorem"

It is consistent that $\mathfrak{b} = \aleph_0$, and there is no countable dominating family.

"Theorem"

It is consistent that $\mathfrak{d} = \aleph_0$.

So we have a definition, which does not solve most of the problems, and no actual theorems. . .

Thank you for your attention!